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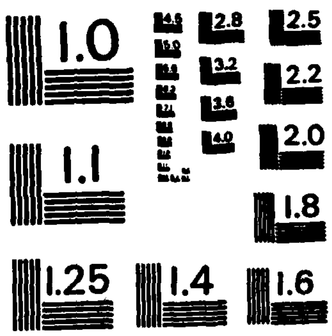

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LIPSCHITZ PROPERTIES OF SOLUTION

IN MATHEMATICAL PROGRAMMING

by

Bernard Cornet and Guy Laroque

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1. Introduction

Let U be an open subset of $\mathbb{R}^k$, V an open subset of $\mathbb{R}^\ell$, f, g_i, $i = 1,\ldots,m$, be mappings from $U \times V$ to $\mathbb{R}$ and C be a nonempty closed convex subset of $\mathbb{R}^m$. For $p = (p_i)$, $g = (g_i)$ in $\mathbb{R}^m$, we denote by $\langle p,q\rangle = \sum_{i=1}^{m} p_i q_i$ the scalar product of $\mathbb{R}^m$ and $||p|| = \sqrt{\langle p,q\rangle}$ the Euclidean norm. For a fixed α in V, we consider the following nonlinear programming problem:

$$
(1.1) \qquad \begin{cases} \text{Find a local minimum of } f(x,\alpha) \\ \text{subject to } g(x,\alpha) \in C \\ \text{and } x \in U, \end{cases}
$$

where $g(x,\alpha) = (g_1(x,\alpha),\ldots,g_m(x,\alpha))$ and x is the variable of the problem.

Assume that the mappings f and g_i, $i = 1,\ldots,m$ are differentiable with respect to the first variable. We denote by $Df(x,\alpha)$, $Dg_i(x,\alpha)$, $i = 1,\ldots,m$ the gradient with respect to x. In this paper, we also consider the following "generalized equation" or "variational inequality" in the unknowns x and λ, which is a necessary condition satisfied by any solution of (1.1) under a constraint qualification assumption:

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$$
(1.2) \qquad \begin{cases} Df(x,\alpha) + \sum_{i=1}^{m} \lambda_i Dg_i(x,\alpha) = 0 \\ \lambda = (\lambda_i) \in N(C, g(x,\alpha)) \end{cases}
$$

where $\lambda = (\lambda_i)$ is a vector of $\mathbb{R}^m$, and $N(C,p)$ denotes the normal cone to C at p:

$$
N(C,p) = \begin{cases} \{\lambda \in \mathbb{R}^m \mid \langle \lambda, p\rangle \geq \langle \lambda, p \rangle \text{ for all } q \text{ in } C\} \text{ if } p \in C \\ \phi \text{ if } p \notin C \end{cases}
$$

In the particular case of equality and/or inequality constraints, i.e., $C = \{0\}^{m_1} \times \mathbb{R}_+^{m_2}$, $m = m_1 + m_2$, the system (1.2) reduces exactly to the Kuhn-Tucker necessary conditions:

$$
(1.3) \qquad \begin{cases} Df(x,\alpha) + \sum_{i=1}^{m} \lambda_i Dg_i(x,\alpha) = 0 \ , \\ g_i(x,\alpha) = 0, \quad i = 1,\ldots,m_1 \ , \\ g_i(x,\alpha) \geq 0, \quad \lambda_i \geq 0, \quad \lambda_i g_i(x,\alpha) = 0 \ , \ i = m_1 + 1,\ldots,m \ . \end{cases}
$$

In this paper, we are concerned with the behaviour of the solutions of problems ~~(1.1), (1.2), (1.3)~~ and also with the Lipschitzian dependence of the set of solutions with respect to the parameter alpha α. The paper is organized as follows. ~~In Section 2, we state~~ the main theorems of the paper are stated. Sufficient conditions are given to have (i) local uniqueness of solutions ~~of (1.1) and (1.2) or (1.3)~~ and (~~ii~~ 2) Lipschitzian dependence of the solution with respect to the parameter alpha α. The proofs of the the theorems are given ~~in Section 3~~ next page

cont

The main tool, in this analysis, is an implicit function theorem for Lipschitzian mappings due to Clarke ~~[1976]~~ (see also Hiriart-Urruty) ~~[1978])~~. The main idea is to write the generalized equation ~~(1.2) or (1.3)~~ as a system of Lipschitzian mapping, to which Clarke's implicit function theorem can be applied. The notion of generalized derivative of the projection mapping on a convex are of particular use.

We end the introduction by indicating briefly the link between this paper and others written previously. The standard implicit-function theorem was used by Fiacco-McCormick [1968], Fiacco [1976] and Robinson [1974] to analyse the sensitivity of perturbed nonlinear programming problems under stronger assumptions than those used in this paper. A general implicit function theorem has been proved by Robinson [1980] for "generalized equation". Applications are given by him to the sensitivity analysis in the case of equality and/or inequality constraints. This corresponds essentially to the results obtained on equation (1.3). The present paper, which generalizes a previous result of the authors (ses also Cornet [1981]) in the case of equality and/or inequality constraints, uses a different technique than the papers quoted above. Another approach for this problem in the convex case is due to Aubin [1981], who uses a different version of the implicit function theorem for convex processes. His result can be derived from ours. Finally we mention the work of Levitin [1975], who has made an analysis of the Lipschitz dependence of the solution of a mathematical programming problem without the use of implicit function theorem.

2. Statement of the Theorems

We posit the following assumption, which describes the general framework of the paper:

Assumption (A.1):

(i) U is an open subset of $\mathbb{R}^k$, V an open subset of $\mathbb{R}$, C a nonempty closed subset of $\mathbb{R}^m$.

(ii) For all α in V, the mappings $f(\cdot,\alpha)$: $U \to \mathbb{R}^m$ and $g_i(\cdot,\alpha)$: $U \to \mathbb{R}$, $i = 1,\dots,m$ are twice continuously differentiable.

(iii) The mappings $g_i(\cdot,\cdot)$: $U \times V \to \mathbb{R}$, $Dg_i(\cdot,\cdot)$: $U \times V \to \mathbb{R}^k$, $i = 1,\dots,m$ and $Df(\cdot,\cdot)$: $U \times V \to \mathbb{R}^k$ are Lipschitzian.

Here, and in the following, we denote by D the operator of differentiation with respect to the first variable x. Without risk of confusion, $Df(x,\alpha)$ will denote, according to the context, either the linear mapping from $\mathbb{R}^k$ to $\mathbb{R}$ or a vector of $\mathbb{R}^k$. Similarly, $D^2f(x,\alpha)$ will denote either the linear mapping from $\mathbb{R}^k$ into $\mathbb{R}^k$, or the matrix of second derivatives of f.

Let $(\bar{x},\bar{\alpha},\bar{\lambda})$ in $U \times V \times \mathbb{R}^m$. We introduce the two following assumptions:

Assumption (A.2): The vectors $Dg_i(\bar{x},\bar{\alpha})$ $i = 1,\dots,m$ are linearly independent.

Assumption (A.3): For all h in $\mathbb{R}^k$, $h \neq 0$, such that $\langle Df(\bar{x},\bar{\alpha}),h\rangle = 0$, then

$$\langle (D^2f(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \bar{\lambda}_i D^2 g_i(\bar{x},\bar{\alpha}))h,h\rangle > 0 \quad .$$

The main theorem of the paper deals with the solution of the system (1.2) when the parameter $\bar{\alpha}$ is slightly perturbed:

Theorem 2.1: Under Assumption (A.1), let $(\bar{x},\bar{\lambda})$ in $U \times \mathbb{R}^k$ satisfy the system of equations (1.2) for $\alpha = \bar{\alpha}$ in V. Assume furthermore that $(\bar{x},\bar{\alpha},\bar{\lambda})$ satisfy (A.2) and (A.3).

Then there exist open neighborhoods U_1 of $\bar{x}$ in U, V_1 of $\bar{\alpha}$ in V, W_1 of $\bar{\lambda}$ in $\mathbb{R}^m$, and mappings $x(\cdot)$: $V_1 \to U_1$, $\lambda(\cdot)$: $V_1 \to W_1$ such that:

(i) the mappings $x(\cdot)$ and $\lambda(\cdot)$ are Lipschitzian;

(ii) $x(\bar{\alpha}) = \bar{x}$, $\lambda(\bar{\alpha}) = \bar{\lambda}$;

(iii) for all α in V_1, $(x(\alpha),\lambda(\alpha))$ is the unique solution in $U_1 \times W_1$ of equation (1.2) at α.

In the case of equality and/or inequality constraints, we are able to weaken assumptions (A.2) and (A.3), getting the strong order conditions of Robinson ([1980], pp. 54-55).

As before, let $(\bar{x},\bar{\alpha},\bar{\lambda})$ in $U \times V \times \mathbb{R}^m$.

Assumption (A.2 bis): The vectors $Dg_i(\bar{x},\bar{\alpha})$ for all $i = 1,\ldots,m$ such that $g_i(\bar{x},\bar{\alpha}) = 0$ are linearly independent.

Assumption (A.3 bis): For all h in $\mathbb{R}^k$, $h \neq 0$, such that

$$\begin{cases} \langle Dg_i(\bar{x},\bar{\alpha}),h\rangle = 0 & i = 1,\ldots,m_1 \\ \langle \bar{\lambda}_i Dg_i(\bar{x},\bar{\alpha}),h\rangle = 0 & \text{for all } i = m_1+1,\ldots,m \\ & \text{such that } g_i(\bar{x},\bar{\alpha}) = 0 \end{cases}$$

Then,

$$<(D^2f(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \bar{\lambda}_i D^2g_i(\bar{x},\bar{\alpha})h,h> > 0 \quad .$$

Note that (A.3) is more restrictive than (A.3 bis), when the first order condition $Df(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \bar{\lambda}_i Dg_i(\bar{x},\bar{\alpha}) = 0$ holds. Note also that the constraints bearing on h for $i = m_1 + 1,\ldots,m$ are not restrictive when $\bar{\lambda}_i = 0$.

<u>Theorem 2.2</u>: Under Assumption (A.1), let $(\bar{x},\bar{\lambda})$ in $U \times \mathbb{R}^k$ satisfy the system of equations (1.3) for $\alpha = \bar{\alpha}$ in V. Assume furthermore that $(\bar{x},\bar{\alpha},\bar{\lambda})$ satisfy (A.2 bis) and (A.3 bis).

Then there exist open neighborhoods U_1 of $\bar{x}$ in U, V_1 of $\bar{\alpha}$ in V, W_1 of $\bar{\lambda}$ in $\mathbb{R}^m$, and mappings $x(\cdot)$: $V_1 \to U_1$, $\lambda(\cdot)$: $V_1 \to W_1$ such that

(i) the mappings $x(\cdot)$ and $\lambda(\cdot)$ are Lipschitzian;

(ii) $x(\bar{\alpha}) = \bar{x}$, $\lambda(\bar{\alpha}) = \bar{\lambda}$;

(iii) for all α in V_1, $(x(\alpha),\lambda(\alpha))$ is the unique solution in $U_1 \times W_1$ of equation (1.3) at α .

Finally, we have to come back to the solution of the initial problem (1.1).

<u>Corollary 2.3</u>: Let $\bar{x}$ in U be a solution of (1.1) at α in V. Assume that the second derivatives $D^2f(x,\alpha)$ and $D^2g_i(x,\alpha)$, $(i = 1,\ldots,m)$, are continuous at $(\bar{x},\bar{\alpha})$.

Then, under the assumptions of Theorem 2.2 (respectively 2.2 in the case of equality and/or inequality constraints), there exist open neighborhoods U_2 of $\bar{x}$ in U_1, V_2 of $\bar{\alpha}$ in V_1 such that the mapping $x(\cdot)$: $V_1 \to U_1$ defined in Theorem 2.1 (respectively 2.2), satisfies the following property:

For all α in V_2, $x(\alpha)$ is the unique local minimum on U_2 of problem (1.1) at α.

3. Proofs

The idea behind the proof of Theorem 2.1 is to write the system (1.2), i.e., the first order necessary conditions, as a system of equations $\phi(x,\lambda,\alpha) = 0$, where the unknown is the couple (x,λ) and ϕ is a locally Lipschitzian mapping. Then we shall get (x,λ) as a function of the parameter α through an implicit function theorem for Lipschitzian mappings which we recall hereafter.

We first recall some definitions. Following Clarke [1975], let ψ be a mapping from an open subset N of $\mathbb{R}^n$ to $\mathbb{R}^n$ and let $\bar{y}$ be an element in N. We define the generalized derivative of ψ at $\bar{y}$ in N denoted $\partial\psi(\bar{y})$, to be the convex hull of the set of matrices M of the form $M = \lim_j D\psi(y^j)$, where y^j is a sequence in N converging to $\bar{y}$ and ψ is differentiable at y^j, with derivative $D\psi(y^j)$. If ψ is locally Lipschitzian, then, by Rademacher's theorem, ψ is differentiable almost everywhere, and one easily shows that the set $\partial\psi(\bar{y})$ is nonempty, convex, and compact. It coincides with the derivative $D\psi(\bar{y})$, whenever ψ is continuously differentiable on a neighborhood of $\bar{y}$.

Now let Φ be a Lipschitzian mapping from an open subset of $\mathbb{R}^n \times \mathbb{R}^\ell$ to $\mathbb{R}^n$. We define the generalized partial derivative of Φ with respect to the first variable at the point $(\bar{y},\bar{\alpha})$, denoted by $\partial_y\Phi(\bar{y},\bar{\alpha})$, as the convex hull of the set of matrices M of the form $M = \lim_j D_y\Phi(y^j,\alpha^j)$, where $\{y^j\}$, $\{\alpha^j\}$ are sequences in $\mathbb{R}^n$ and $\mathbb{R}^\ell$ converging to $\bar{y}$ and $\bar{\alpha}$ respectively, Φ is differentiable at (y^j,α^j) and $D_y\Phi(y^j,\alpha^j)$ denotes the usual partial Jacobian matrix with respect to the first variable y.

We now state the implicit function theorem for Lipschitzian mappings due to Clarke [1976] (see also Hiriart-Urruty [1978]).

<u>Theorem 3.1</u>: Let E, F be finite dimensional real vector spaces, let Ω be an open subset of $E \times F$ and let ϕ be a Lipschitzian mapping from Ω to E. We suppose that, at a point $(\bar{y},\bar{\alpha})$ in Ω, the generalized partial derivative $\partial_y\phi(\bar{y},\bar{\alpha})$ has maximal rank (in the sense that every matrix in $\partial_y\phi(\bar{y},\bar{\alpha})$ has maximal rank) and that $\phi(\bar{y},\bar{\alpha}) = 0$. Then, there exist open neighborhoods U of $\bar{y}$, V of $\bar{\alpha}$ and a Lipschitzian mapping f from V to U, such that:

(i) $\bar{y} = f(\bar{\alpha})$

(ii) for all α in V, $f(\alpha)$ is the unique element in U satisfying $\phi(f(\alpha),\alpha) = 0$.

Before proceeding to the proof of Theorem 2.1, we need some further definitions, as well as two lemmas. Let C be a nonempty closed convex subset of $\mathbb{R}^m$ and p an element of $\mathbb{R}^m$. We denote by

$\pi(p)$ the projection of p on C. We recall that $\pi(p)$ is uniquely defined by one of the two equivalent assertions:

$$(3.2) \qquad \pi(p) \in C \text{ and } ||\pi(p) - p|| \leq ||\pi(p) - q|| \text{ for all } q \text{ in } C ;$$

$$(3.3) \qquad \pi(p) \in C \text{ and } \langle\pi(p) - p, \pi(p) - q\rangle \leq 0 \text{ for all } q \text{ in } C .$$

Furthermore, the mapping $p \to \pi(p)$, from $\mathbb{R}^m$ to $\mathbb{R}^m$ is Lipschitzian of constant 1 (i.e., for all p, q in $\mathbb{R}^m$, $||\pi(p) - \pi(q)|| \leq ||p - q||$, see for example, Rockafellar [1970]).

<u>Lemma 3.4</u>: Let C be a nonempty, closed, convex subset of $\mathbb{R}^m$, π: $\mathbb{R} \to \mathbb{R}^m$ the projection mapping on C. Then, for every p in $\mathbb{R}^m$, and every h in $\mathbb{R}^m$.

(i) $\langle\Delta h, h - \Delta h\rangle \geq 0$ for every Δ in $\partial\pi(p)$;

(ii) $\langle p - \pi(p), \Delta h\rangle = 0$ for every Δ in $\partial\pi(p)$.

<u>Proof:</u>

(i) By (3.3), for all p, q in $\mathbb{R}^m$, $\langle p - \pi(p), \pi(q) - \pi(p)\rangle \leq 0$ and $\langle q - \pi(q), \pi(p) - \pi(q)\rangle \leq 0$. Summing up the two inequalities, taking $q = p + th$, for $t > 0$, in in $\mathbb{R}^m$, dividing by t^2, we get:

$$(3.5) \qquad \left\langle \frac{\pi(p + th) - \pi(p)}{t}, h \right\rangle \geq \left|\left|\frac{\pi(p + th - \pi(p)}{t}\right|\right|^2 .$$

Let Δ in $\partial\pi(p)$. From the definition of $\partial\pi(p)$, there exist weights $W^i > 0$, $i = 1,\ldots,I$, $\sum_{i=1}^{I} W^i = 1$ and $(m \times m)$ matrices Δ^i, $i = 1,\ldots,I$ in $\partial\pi(p)$ such that $\Delta = \sum_{i=1}^{I} W^i\Delta^i$ and the Δ^i satisfy the following property. For all i, there exists a sequence $\{p^{ij}\}$ in

$\mathbb{R}^m$ converging to p, such that the derivative $D\pi(p^{ij})$ exists and $\Delta^i = \lim\limits_j D\pi(p^{ij})$. Using (3.5) with $p = p^{ij}$, when $t \to 0$, $j \to \infty$, we get at the limit, for all $i = 1,\ldots,I$:

$$<\Delta^i h, h> \geq ||\Delta^i h||^2 \quad .$$

Using the fact that the mapping $p \to ||p||^2$ is convex, we get:

$$||\Delta h||^2 - <\Delta h,h> \leq \sum_{i=1}^{I} W^i(||\Delta^i h||^2 = <\Delta^i h,h>) \leq 0 \quad .$$

(ii) From (3.3), taking $q = \pi(p + th)$, h in $\mathbb{R}^m$, $t \neq 0$, dividing by t, we get:

$$<p = \pi(p), \frac{\pi(p + th) - \pi(p)}{t}> \leq 0 \quad \text{if} \quad t > 0$$

$$<p - \pi(p), \frac{\pi(p + th) - \pi(p)}{t}> \geq 0 \quad \text{if} \quad t < 0 \quad .$$

Thus, at a point p in $\mathbb{R}^m$ where π is differentiable, when $t \to 0$ at the limit in the two above inequalities, we get:

$$<p - \pi(p), D\pi(p)h> = 0 \quad .$$

Now let p be an arbitrary point in $\mathbb{R}^m$ and let Δ be in $\partial\pi(p)$. Using the definition of $\partial\pi(p)$, as in the proof of the first part of the lemma, we get $<p - \pi(p), \Delta h> = 0$. This ends the proof of Lemma 3.4.

<u>Remark 3.5</u>: When $C = \{0\}^{m_1} \times \mathbb{R}_+^{m_2}$ with $m_1 + m_2 = m$, it is easy to compute $\partial\pi(p)$ for all p in $\mathbb{R}^m$. Let Δ_{ij} be the element of row i and column j of a $m \times m$ matrix . Then Δ belongs to $\partial\pi(p)$ if and only if:

$$
\begin{array}{lll}
\Delta_{ij} = 0 & \text{for all } i \neq j & \\
\Delta_{ii} = 0 & \text{for } i = 1,\dots,m_1 & \\
\left.\begin{array}{ll}
\Delta_{ii} = 1 & \text{if } p_i > 0 \\
0 \le \Delta_{ii} \le 1 & \text{if } p_i = 0 \\
\Delta_{ii} = 0 & \text{if } p_i < 0
\end{array}\right\} & & \text{for } i = m_1 + 1,\dots,m_2
\end{array}
$$

Lemma 3.6: Let C be a nonempty, closed, convex subset of $\mathbb{R}^m$, let π: $\mathbb{R}^m \to \mathbb{R}^m$ be the projection mapping on C, let p be an element in C and λ be in $\mathbb{R}^m$. Then the following assertions are equivalent:

(i) $\lambda \in N(C,p)$, i.e., $\langle\lambda,p\rangle \geq \langle\lambda,q\rangle$ for all q in C ;

(ii) there exists μ in $\mathbb{R}^m$ such that $p = \pi(\mu)$ and $\lambda = \mu - \pi(\mu)$.

Proof: (i)$\to$(ii). Let $\lambda \in N(C,p)$ and $\mu = \lambda + p$. It suffices to show that $p = \pi(\mu)$. Since λ belongs to $N(C,p)$, for all q in C, $\langle\mu - p, q - p\rangle \leq 0$. By (3.3), we deduce $p = \pi(\mu)$. (ii)$\to$(i): Let μ in $\mathbb{R}^m$ with $p = \pi(\mu)$ and $\lambda = \mu - \pi(\mu)$. By (3.3), for all q in C, $\langle\pi(\mu) - \mu, \pi(\mu) - q\rangle \leq 0$. Therefore, for all q in C, $\langle\lambda,p\rangle \geq \langle\lambda,q\rangle$ and λ belongs to $N(C,p)$.

This completes the proof of Lemma 3.6.

In order to prove Theorem 2.1, we modify the necessary conditions (1.2) by introducing an auxiliary variable μ.

Let (x,λ) be in $U \times \mathbb{R}^m$, α in V. From Lemma 3.6, (x,λ) satisfies the first order condition (1.2) at α:

$$(1.2) \qquad \begin{cases} Df(x,\alpha) + \sum_{i=1}^{m} \lambda_i Dg_i(x,\alpha) = 0 \\ \lambda \in N(C, g(x,\alpha)) \end{cases}$$

if and only if there exists μ in $\mathbb{R}^m$ such that (x,λ,μ) satisfies the system of equations:

$$(3.7) \qquad \begin{cases} Df(x,\alpha) + \sum_{i=1}^{m} \lambda_i Dg_i(x,\alpha) = 0 \\ g(x,\alpha) - \pi(\mu) = 0 \\ g(x,\alpha) + \lambda - \mu = 0 \ . \end{cases}$$

Let $y = (x,\lambda,\mu)$. We denote by $\phi(y,\alpha)$ the left-hand side of the system (3.7). Φ is a mapping from an open subset of $\mathbb{R}^{k+2m} \times \mathbb{R}^{\ell}$ into $\mathbb{R}^{k+2m}$. Since the projection mapping π on C is Lipschitzian, ϕ is locally Lipschitzian.

<u>Proof of Theorem 2.1</u>: Let $\bar{\mu} = \bar{\lambda} + g(\bar{x},\bar{\alpha})$, and $\bar{y} = (\bar{x},\bar{\lambda},\bar{\mu})$. By Clarke's implicit function theorem (Theorem 3.1), it suffices to prove that the generalized partial derivative $\partial_y \phi(\bar{y},\bar{\alpha})$ is of maximal rank.

Let Δ be an arbitrary element in $\partial\pi(\bar{\mu})$. Then $\partial_y \phi(\bar{y},\bar{\alpha})$ is made up of the matrices of the following form:

$$M = \begin{bmatrix} D^2f(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \bar{\lambda}_i D^2 g_i(\bar{x},\bar{\alpha}) & Dg(\bar{x},\bar{\alpha})^T & 0 \\ Dg(\bar{x},\bar{\alpha}) & 0 & -\Delta \\ Dg(\bar{x},\bar{\alpha}) & I_m & -I_m \end{bmatrix}$$

where $Dg(\bar{x},\bar{\alpha})$ denotes the $m \times k$ matrix of rows $Dg_i(\bar{x},\bar{\alpha})$, $i = 1,\dots,m$, $Dg(\bar{x},\bar{\alpha})^T$ its transpose, and I_m the $(m \times m)$ identity matrix.

To show that M is of full rank, let there be vectors a in $\mathbb{R}^k$, b and c in $\mathbb{R}^m$ such that:

$$(3.8) \qquad M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

It suffices to show that $a = 0$ and $b = c = 0$.

Replacing M by its value in (3.8), we get:

$$(3.9) \qquad \begin{cases} [D^2f(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \bar{\lambda}_i D^2 g_i(\bar{x},\bar{\alpha})]a + Dg(\bar{x},\bar{\alpha})^T b = 0 \\ Dg(\bar{x},\bar{\alpha})a - \Delta c = 0 \\ Dg(\bar{x},\bar{\alpha})a + b - c = 0 \quad . \end{cases}$$

Premultiplying the first equation by a, using the other ones and Lemma 3.4 (i), we get:

$$a^T[D^2f(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \bar{\lambda}_i D^2 g_i(\bar{x},\bar{\alpha})]a = -a^T Dg(\bar{x},\bar{\alpha})^T b$$

$$= -b^T \Delta c$$

$$= -(c - \Delta c)^T \Delta c \leq 0 \quad .$$

On the other hand, using the first order condition (2.2) and the above equations, we get:

$$Df(\bar{x},\bar{\alpha})a = -\bar{\lambda}^T Dg(\bar{x},\bar{\alpha})a = -\bar{\lambda}^T \Delta c$$

Since $\bar{\lambda} = \bar{\mu} - \pi(\bar{\mu})$ and Δ belongs to $\partial\pi(\bar{\mu})$, by Lemma (3.4) (ii), we obtain:

$$(3.10) \qquad Df(\bar{x},\bar{\alpha})a = 0 \quad .$$

By Assumption (A.3) of Theorem 2.1, we get $a = 0$. The first line of (3.9) implies then $Dg(\bar{x},\bar{\alpha})^T b = 0$. Since by Assumption (A.2) the vectors $Dg_i(\bar{x},\bar{\alpha})$, $i = 1,\ldots,m$ are independent, we have $b = 0$. Thus, $c = 0$. This ends the proof of Theorem 2.1.

<u>Proof of Theorem 2.2</u>: It follows the same line as that of Theorem 2.1. First note that we can ignore the constraints i in $\{m_1 + 1,\ldots,m\}$ such that $g_i(\bar{x},\bar{\alpha}) > 0$. In fact, the continuity of the solution $x(\alpha)$ which we are looking for implies that these constraints will stay unbinding. The only difference lies then in the study of system (3.9) where Remark 3.5 and Assumption (A.3 bis) are used.

In fact, the second line in (3.9), using Remark 3.5, leads to:

$$Dg_i(\bar{x},\bar{\alpha})a = 0 \quad \text{for} \quad i = 1,\ldots,m_1 \quad .$$

Furthermore, consider an i in $\{m_1 + 1,\ldots,m\}$ such that $g_i(\bar{x},\bar{\alpha}) = 0$ and $\bar{\lambda}_i > 0$. Then by (3.7), $\bar{\mu}_i < 0$. Thus by Remark 3.5, $\Delta_{ii} = 0$, therefore:

$$Dg_i(\bar{x},\bar{\alpha})a = 0 \quad .$$

Consequently, in the proof of Theorem 2.1, we can replace (3.10) by:

$$(3.11) \qquad \begin{array}{ll} Dg_i(\bar{x},\bar{\alpha})a = 0 & \text{for } i = 1,\ldots,m_1 \\ \bar{\lambda}_i Dg_i(\bar{x},\bar{\alpha})a = 0 & \text{for all } i \text{ in } \{m_1 + 1,\ldots,m\} \\ & \quad \text{such that } g_i(\bar{x},\bar{\alpha}) = 0 \quad . \end{array}$$

By Assumption (A.3 bis), we get $a = 0$. The rest of the proof is unchanged. This ends the proof of Theorem 2.2.

Proof of Corollary 2.3: By Robinson [1980], under Assumption (A.2), the system of equations (1.2) (respectively (1.3)) is a necessary condition for the solution of (1.1). Let $x(\cdot)$: $V_1 \to U_1$ be the mapping defined in Theorem 2.1 (respectively 2.2). Then, for all α in V_1, $x(\alpha)$ is the only candidate for a local minimum on U_1, for problem (1.1).

To show that it corresponds in fact to a local minimum, we have to check that the sufficient conditions (A.3) (respectively (A.3 bis)) are satisfied in an open neighborhood of $(\bar{x},\bar{\alpha})$ (see Robinson [1980] for a proof that (A.3) and (A.3 bis) are indeed sufficient conditions).

We prove it for (A.3), leaving the other similar case to the reader. Suppose on the contrary that for a sequence $\{(x^j,\alpha^j)\}$ converging to $(\bar{x},\bar{\alpha})$, (A.3) is violated at $(x^j,\alpha^j,\lambda(\alpha^j))$. There exists h^j in $\mathbb{R}^k$, $||h^j|| = 1$ such that $\langle Df(x^j,\alpha^j),h^j\rangle = 0$ and $\langle (D^2f(x^j,\alpha^j) + \sum_{i=1}^{m} \lambda_i(\alpha^i)D^2g_i(x^j,\alpha^j))h^j,h^j\rangle \leq 0$ where $\lambda(\alpha)$ is the Lipschitzian function defined in Theorem 2.1. Since the h^j stay in a compact, there exists a converging subsequence towards $\bar{h}$. From the continuity assumptions, we have $\langle Df(\bar{x},\bar{\alpha}),\bar{h}\rangle = 0$ and

$$\langle (D^2f(\bar{x},\bar{\alpha}) + \sum_{i=1}^{m} \lambda_i(\bar{\alpha})D^2g_i(\bar{x},\bar{\alpha}))\bar{h},\bar{h}\rangle \leq 0 \quad .$$

Hence this contradicts Assumption (A.3). This ends the proof of Corollary 2.3.

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